Lorentz transformations from the first postulate

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We present in this paper a derivation of the Lorentz transformation by invoking the principle of relativity alone, without resorting to the a priori assumption of the existence of a universal limiting velocity. Such a velocity is shown to be a necessary consequence of the first postulate, and the fact that it is not infinite is borne out by experiment.

I. INTRODUCTION

In the course of teaching special relativity to undergraduates, the authors have rediscovered the fact that it is possible to derive the Lorentz transformations from the first of Einstein's two postulates alone. That in fact it is not necessary at all to postulate the concept of a universal limiting velocity nor in fact to identify it with the speed of light has been known for a long time.[1-6] Nevertheless, these works have not received sufficient emphasis in student texts on relativity despite the fact that they allow a much more acceptable philosophical view point to be adopted than is possible in any of the numerous "two postulate" derivations. The transformation derived has the exact form of the Lorentz transformation with the exception that the velocity of light c is replaced by a univversal constant σ which is not necessarily the same as c. The Galilean transformation (the case of $\sigma=\infty$) is ruled out by experimental evidence.

In view of the general interest shown through the number of papers which have appeared in this Journal in the last few years[7-11] outlining new ways of teaching the subject, it does not seem out of place to present here a completely alternative derivation which has the intrinsic merit that it avoids any *a priori* mention of limiting velocities. Our argument rests only on the simple notion of the isotropy and homogeneity of space and time together with the relativity postulate. Because our mathematical deductions do not involve the obscurities of previous authors, but require only the barest minimum of matrix theory and algebra, we feel that they should be easily understood by undergraduate students. In the following, we present our derivation followed by some relevant discussions in Sec. III.

II. DERIVATION OF LORENTZ TRANSFORMATIONS

Assuming the existence of classical inertial frames, let S and S' denote any two such frames in relative motion. We further assume the usual homogeneity of time, and homogeneity and isotropy of space. This permits us to conclude that there is no loss of physical generality through the choice of the mathematically simple situation of relative motion for which the spatial axes of S and S' coincide at $t = t' = 0$, and for which the spatial origin of S' moves with velocity v along the z axis of S.

At this point, we take for granted that all the well known arguments have been given that lead up to the linear transformation equations

$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= \alpha z + \beta t \\ t' &= \gamma z + \delta t, \end{aligned} \tag{1}$$

relating the coordinates of arbitrary events, in S and S' via constants α, β, γ, δ which can only depend on v.[12] It is emphasized that none of the arguments require the second postulate.

We further reduce the number of unknown constants in the usual way from four to two by requiring that all fixed spatial points in S' (for simplicity, the origin) have velocity v relative to S, and that all fixed spatial points in S have velocity $-v$ relative to S'.[13] Then the nontrivial part of the transformation [Eq. (1)] may be written in the matrix form

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \alpha & -v\alpha \\ \gamma & \alpha \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix} \tag{2}$$

The two unknown functions $\alpha(v)$ and $\gamma(v)$ may be shown to be even and odd functions of v respectively [i.e., $\alpha(v)=\alpha(-v)$ and $\gamma(v)=-\gamma(-v)$]. To see this, we switch the directions of both the z and z' axes (equivalent to reflection in the xy plane), and write

$$z_{\rm re} \equiv -z, \quad z_{\rm re}' \equiv -z'.$$

Since $S_{\rm re}'$ now moves with velocity $-v$ relative to $S_{\rm re}$, we have

$$\begin{pmatrix} z_{\rm re}' \\ t' \end{pmatrix} = \begin{pmatrix} \alpha(-v) & -(-v)\alpha(-v) \\ \gamma(-v) & \alpha(-v) \end{pmatrix} \begin{pmatrix} z_{\rm re} \\ t \end{pmatrix}. \tag{3}$$

However, by merely changing the sign of z and z' in Eq. (2) we have the alternative equation

$$\begin{pmatrix} z_{\rm re}' \\ t' \end{pmatrix} = \begin{pmatrix} \alpha(v) & +v\alpha(v) \\ -\gamma(v) & \alpha(v) \end{pmatrix} \begin{pmatrix} z_{\rm re} \\ t \end{pmatrix}. \tag{4}$$

Comparison of Eqs. (3) and (4) leads to the required conclusion.[14] Clearly for $v=0$, the physical meaning of the transformation demands that

$$\alpha(0)=1 \quad \text{and} \quad \gamma(0)=0,$$

requirements which are consistent with α and γ being even and odd functions of v respectively.

To evaluate α and γ, we now utilize the matrix properties of the transformation. Thus, using the notation $S \xrightarrow{v} S'$ to signify that S' moves with velocity v relative to S', we consider the following sequence and its transformation matrix:

$$S \xrightarrow{v_2} S' \xrightarrow{v_1} S''$$

$$\begin{pmatrix} \alpha_1 & -v_1\alpha_1 \\ \gamma_1 & \alpha_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & -v_2\alpha_2 \\ \gamma_2 & \alpha_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1\alpha_2 - v_1\alpha_1\gamma_2 & -\alpha_1\alpha_2(v_1+v_2) \\ \alpha_1\alpha_2 + \alpha_1\gamma_2 & \alpha_1\alpha_2 - v_2\gamma_1\alpha_2 \end{pmatrix} \tag{5}$$

where we have used the shorthand notation

$$\alpha_i \equiv \alpha(v_i) \quad \text{and} \quad \gamma_i \equiv \gamma(v_i),$$

However, since S and S'' are both inertial, there must exist a single velocity v_0 which relates their motions, and hence a single transformation of the form:

$$S \xrightarrow{v_0} S''$$

$$\begin{pmatrix} \alpha_0 & -v_0\alpha_0 \\ \gamma_0 & \alpha_0 \end{pmatrix}. \tag{6}$$

Since Eqs. (5) and (6) represent the same transformation, and must therefore be similar in form, we immediately obtain by equating the diagonal elements of Eq. (5) that

$$\gamma_1/v_1\alpha_1 = \gamma_2/v_2\alpha_2 \, , \tag{7}$$

and purely from the fact that the LHS of Eq. (7) is a function of v_1 only while the RHS is a function of v_2 only, we conclude that

$$\gamma(v)/v\alpha(v) = k, \tag{8}$$

where k is a universal constant not dependent on v. The transformations of Eqs. (5) and (6) now read

$$S \xrightarrow{v_2} S' \xrightarrow{v_1} S''$$

$$\alpha_1\alpha_2(1-kv_1v_2)\begin{bmatrix} 1 & -\dfrac{v_2+v_2}{1-kv_1v_2} \\ \dfrac{v_1+v_2}{1-kv_1v_2} & 1 \end{bmatrix}, \tag{9}$$

and

$$S \xrightarrow{v_0} S''$$

$$\alpha_0 \begin{bmatrix} 1 & -v_0 \\ kv_0 & 1 \end{bmatrix}. \tag{10}$$

By equating corresponding elements in Eqs. (9) and (10), we can immediately identify v_0, and thereby obtain the velocity addition law

$$v_0 = (v_1+v_2)/(1-kv_1v_2), \tag{11}$$

and the transformation of $\alpha(v)$:

$$\alpha(v_0) = \alpha(v_1)\alpha(v_2)(1-kv_1v_2). \tag{12}$$

Now regardless of the value of k, one may always choose a real number v such that by setting

$$v_1 = -v_2 = v,$$

the denominator in Eq. (11) remains finite, and so leads to $v_0=0$. This is equivalent to the successive forward and inverse transformations

$$S \xrightarrow{v} S' \xrightarrow{-v} S.$$

For this case, using $\alpha(v_0=0)=1$, we obtain from Eq. (12):

$$\alpha(v)\alpha(-v)(1+kv^2)=1.$$

Thus,

$$\alpha(v) = (1+kv^2)^{-1/2} \tag{13}$$

follows from the even nature of $\alpha(v)$. The negative sign in front of the square root is automatically excluded by the condition $\alpha(0)=1$.

It now remains to show that k can only take zero or negative real values. For if k is positive, the expression [Eq. (11)] leads to the physically absurd result that two

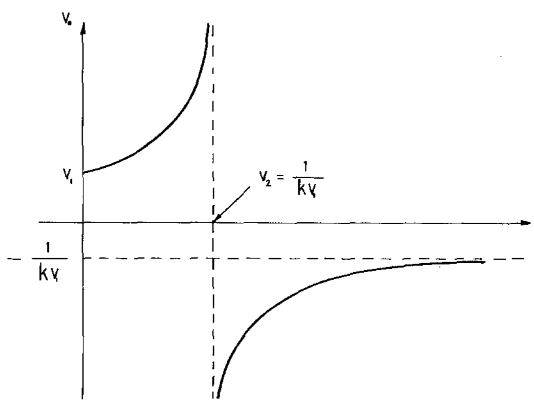

Fig. 1. Velocity addition for positive k [Eq. (11)]. Resultant velocity v_0 is plotted as a function of v_2 with v_1 fixed (and positive).

velocities in the same direction may add to a velocity in the reverse direction. We note that for positive k, Eq. (13) does not impose any restriction on the velocity v attainable by physical objects, so that the condition $kv_1v_2>1$ can in principle always be satisfied. As an aid to students, it may be instructive to plot qualitatively v_0 as a function of v_2 with v_1 fixed (and positive, say) for Eq. (11).

Figure 1 shows clearly that a positive k violates our physical intuition about velocity addition. We see that the resultant velocity v_0 increases to infinity at $v_2=1/kv_1$, at which point velocity reversal takes place.

The mathematically trivial possibility $k=0$ simply leads to the familiar Galilean velocity addition and transformation laws,

$$v_0 = v_1 + v_2$$

$$z' = z - vt$$

$$t' = t$$

while the Lorentz transformations follow on putting $k=-1/\sigma^2$, with σ a positive constant:

$$\alpha(v) = (1 - v^2/\sigma^2)^{-1/2}$$

and

$$\gamma(v) = (-v/\sigma^2)(1 - v^2/\sigma^2)^{-1/2}.$$

The Einstein formula for parallel velocity addition also results [from Eq. (11)]:

$$v_0 = \frac{v_1 + v_2}{1 + v_1 v_2/\sigma^2} \quad \text{with} \quad 0 < \sigma^2 < \infty.$$

It follows that σ plays the role of a limiting universal speed, which though unique, is as yet arbitrary, and need not be identified with the speed of light. The nature of the Lorentz formulas dictate that no particle may exceed this limiting speed without leading to imaginary values for the transformation coefficients α and γ. We emphasize, however, that this universal speed was *not* assumed in deriving the formulas, but follows as a consequence from them. The possibility that $\sigma^2=\infty$ could not be ruled out on the basis of theory alone, but there is an abundance of experimental evidence which points to the fact that σ^2 is finite. Among these are experiments on the relativistic increase in mass of elementary particles with velocity (e.g., electrons) the limiting electron speeds observed in linear accelerators,[11] and the increase in mean life time of unstable high energy particles (such as mesons) in flight in accordance with the time-dilation formula which is easily derivable from the Lorentz transformation. The fact that the mass of a particle increases with its speed at all is an indication that the Galilean transformations ($\sigma^2=\infty$) are invalid. Experiments actually show that the form of dependence of mass on velocity follow closely that derived from the Lorentz transformation, and that the limiting velocity σ is indistinguishable from the speed of light c to within present experimental limits of accuracy. The basis of special relativity, therefore, appears to rest on the first postulate alone, together with the crucial recognition that space and time are not absolute, a concept which is foreign to Newtonian mechanics. That the limiting speed σ is finite is an experimental fact.

The method presented above derives jointly the Lorentz coefficients and the velocity addition law without assuming the concept of a universal speed. Rather, such a concept is *discovered* in the process of derivation. Furthermore, the derivation allows one to ask the question whether or not light as a physical phenomenon possesses the limiting speed σ, rather than state it as a postulate in the way of Einstein. Clearly such a question may be meaningfully asked since the universal constant σ can be determined from experiments not involving light. Such a standpoint has proved philosophically satisfying to students, and this is not diminished by the empirical fact that σ and c (the local velocity of light *in vacuo*) are as yet experimentally indistinguishable. Indeed, the possibility that the photon should possess a nonzero rest mass μ_0 (first suggested by de Broglie[15-17]) cannot be lightly dismissed. This would necessitate that $c < \sigma$, however small the discrepancy. Recent estimates[18] place an upper limit of $\mu_0 \leqslant 4 \times 10^{-48}$ g to this rest mass. The corresponding speed for a photon in the optical region would be $c > (1 - 10^{-20})\sigma$, suggesting a departure of less than $10^{-20}\alpha$ from the limiting speed σ, a departure which would be exceedingly difficult to detect with presently available techniques. Nevertheless, the implications cannot be ignored.

To round off this discussion, we mention that in giving coordinates to events in inertial systems by the usual method of standard clocks and signals, it is not imperative to conceive of such a signal as being light. Any other signal which is *believed* to travel with a characteristic uniform speed in the rest frame of the source will also suffice. As an example, one can think of each inertial observer equipped with a standard gun, firing standard bullets with the aid of which inertial clocks spread out in space may be synchronized. Electromagnetic waves are generally considered the most suitable signal simply because they do not require a material medium for their transmission, and because their speed in vacuum is independent of their frequency, intensity, or direction of propagation.[12] The fact that they propagate with the highest speed known experimentally is obviously of practical value, and the invariance of this speed demanded by the second postulate has given it added theoretical significance. Our approach, however, does not require any preconceived notion of special signals. Instead, the Lorentz transformations are shown to be a manifestation of the properties of space-time of inertial systems. The concept of a limiting signal speed arises only as a consequence.

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[6]For an extensive bibliography, refer to H. Arzelies, *Relativistic Kinematics* (Pergamon, New York, 1966). This author adopts a viewpoint which appears to be contrary to the spirit of the present

paper in that he regards "the properties of light [to be] so peculiar that it is perhaps reasonable to consider it as a fundamental phenomenon [rather] than as a special case." In our view, however, before one can even talk of properties of light, "peculiar" or otherwise, one ought to have a philosophical concept of space–time which is independent of light as a phenomenon. See also de Broglie (Refs. 15–17).

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[13]This procedure is adopted from Moller, Ref. 12. By considering the motion of the origin O' of S' ($z' = 0$), we obtain from Eq. (1): $v = z/t = -\beta/\alpha$. From the motion of the origin O of S ($z = 0$) we have $-v = z'/t' = \beta/\alpha$. Hence, $\alpha = \delta$ and $\beta = -v\alpha$.

[14]The same conclusion can be reached by considering a reversal of the t axis.

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THE OVERPRODUCTION OF TRUTH

The man of knowledge in our time is bowed under a burden he never imagined he would ever have: the overproduction of truth that cannot be consumed. For centuries man lived in the belief that truth was slim and elusive and that once he found it the troubles of mankind would be over. And here we are in the closing decades of the 20th century, choking on truth. There has been so much brilliant writing, so many genial discoveries, so vast an extension and elaboration of these discoveries—yet the mind is silent as the world spins on its age-old demonic career.

—Ernest Becker, *The Denial of Death*
(The Free Press, New York, 1973)